

Foundations of machine learning

Local asymptotic Normality

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Winter 2026

The Normal means model as asymptotic approximation

- The Normal means model might seem quite special.
- But asymptotically, any sufficiently smooth parametric model is equivalent.
- Formally: The likelihood ratio process of n i.i.d. draws Y_i from the distribution

$$P_{\theta_0+h/\sqrt{n}}^n,$$

converges to the likelihood ratio process of one draw X from

$$N\left(h, I_{\theta_0}^{-1}\right)$$

- Here h is a local parameter for the model around θ_0 , and I_{θ_0} is the Fisher information matrix.

- Suppose that P_θ has a density f_θ relative to some measure.
- Recall the following definitions:
 - Log-likelihood: $\ell_\theta(Y) = \log f_\theta(Y)$
 - Score: $\dot{\ell}_\theta(Y) = \partial_\theta \log f_\theta(Y)$
 - Hessian $\ddot{\ell}_\theta(Y) = \partial_\theta^2 \log f_\theta(Y)$
 - Information matrix: $I_\theta = \text{Var}_\theta(\dot{\ell}_\theta(Y)) = -E_\theta[\ddot{\ell}_\theta(Y)]$

- Likelihood ratio process:

$$\prod_i \frac{f_{\theta_0+h/\sqrt{n}}(Y_i)}{f_{\theta_0}(Y_i)},$$

where $Y_1, \dots, Y_n$ are i.i.d. $P_{\theta_0+h/\sqrt{n}}$ distributed.

Practice problem (Taylor expansion)

- Using this notation, provide a second order Taylor expansion for the log-likelihood $\ell_{\theta_0+h}(Y)$ with respect to h .
- Provide a corresponding Taylor expansion for the log-likelihood of n i.i.d. draws Y_i from the distribution $P_{\theta_0+h/\sqrt{n}}$.
- Assuming that the remainder is negligible, describe the limiting behavior (as $n \rightarrow \infty$) of the log-likelihood ratio process

$$\log \prod_i \frac{f_{\theta_0+h/\sqrt{n}}(Y_i)}{f_{\theta_0}(Y_i)}.$$

Solution

- Expansion for $\ell_{\theta_0+h}(Y)$:

$$\ell_{\theta_0+h}(Y) = \ell_{\theta_0}(Y) + h' \cdot \dot{\ell}_{\theta_0}(Y) + \frac{1}{2} \cdot h \cdot \ddot{\ell}_{\theta_0}(Y) \cdot h + \textit{remainder}.$$

- Expansion for the log-likelihood ratio of n i.i.d. draws:

$$\log \prod_i \frac{f_{\theta_0+h'/\sqrt{n}}(Y_i)}{f_{\theta_0}(Y_i)} = \frac{1}{\sqrt{n}} h' \cdot \sum_i \dot{\ell}_{\theta_0}(Y_i) + \frac{1}{2n} h' \cdot \sum_i \ddot{\ell}_{\theta_0}(Y_i) \cdot h + \textit{remainder}.$$

- Asymptotic behavior (by CLT, LLN):

$$\begin{aligned} \Delta_n &:= \frac{1}{\sqrt{n}} \sum_i \dot{\ell}_{\theta_0}(Y_i) \rightarrow^d N(0, I_{\theta_0}), \\ \frac{1}{2n} \cdot \sum_i \ddot{\ell}_{\theta_0}(Y_i) &\rightarrow^p -\frac{1}{2} I_{\theta_0}. \end{aligned}$$

- Suppose the remainder is negligible.
- Then the previous slide suggests

$$\log \prod_i \frac{f_{\theta_0+h/\sqrt{n}}(Y_i)}{f_{\theta_0}(Y_i)} =^A h' \cdot \Delta - \frac{1}{2} h' I_{\theta_0} h,$$

where

$$\Delta \sim N(0, I_{\theta_0}).$$

- Theorem 7.2 in van der Vaart (2000), chapter 7 states sufficient conditions for this to hold.
- We show next that this is the same likelihood ratio process as for the model

$$N\left(h, I_{\theta_0}^{-1}\right).$$

Practice problem

- Suppose $X \sim N(h, I_{\theta_0}^{-1})$
- Write out the log likelihood ratio

$$\log \frac{\phi_{I_{\theta_0}^{-1}}(X - h)}{\phi_{I_{\theta_0}^{-1}}(X)}.$$

Solution

- The Normal density is given by

$$\varphi_{I_{\theta_0}^{-1}}(x) = \frac{1}{\sqrt{(2\pi)^k |\det(I_{\theta_0}^{-1})|}} \cdot \exp\left(-\frac{1}{2}x' \cdot I_{\theta_0} \cdot x\right)$$

- Taking ratios and logs yields

$$\log \frac{\varphi_{I_{\theta_0}^{-1}}(X - h)}{\varphi_{I_{\theta_0}^{-1}}(X)} = h' \cdot I_{\theta_0} \cdot x - \frac{1}{2}h' \cdot I_{\theta_0} \cdot h.$$

- This is exactly the same process we obtained before, with $I_{\theta_0} \cdot X$ taking the role of Δ .

Why care

- Suppose that $Y_i \sim^{iid} P_{\theta+h/\sqrt{n}}$, and $T_n(Y_1, \dots, Y_n)$ is an arbitrary statistic that satisfies

$$T_n \rightarrow^d L_{\theta,h}$$

for some limiting distribution $L_{\theta,h}$ and all h .

- Then $L_{\theta,h}$ is the distribution of some (possibly randomized) statistic $T(X)$!
- This is a (non-obvious) consequence of the convergence of the likelihood ratio process.
- cf. Theorem 7.10 in van der Vaart (2000).

Maximum likelihood and shrinkage

- This result applies in particular to $T =$ estimators of θ .
- Suppose that $\hat{\theta}^{ML}$ is the maximum likelihood estimator.
- Then $\hat{\theta}^{ML} \rightarrow^d X$, and any shrinkage estimator based on $\hat{\theta}^{ML}$ converges in distribution to a corresponding shrinkage estimator in the limit experiment.

References

van der Vaart, A. W. (2000). Asymptotic statistics. Cambridge University Press, chapter 7.

Hansen, B. E. (2016). Efficient shrinkage in parametric models. Journal of Econometrics, 190(1):115–132.