

Foundations of machine learning
Reinforcement learning

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Outline

- Markov decision problems: Goal oriented interactions with an environment.
- Expected updates – dynamic programming.
Familiar from economics. Requires complete knowledge of transition probabilities.
- Sample updates: Transition probabilities are unknown.
 - On policy: Sarsa.
 - Off policy: Q-learning.
- Approximation: When state and action spaces are complex.
 - On policy: Semi-gradient Sarsa.
 - Off policy: Semi-gradient Q-learning.
 - Deep reinforcement learning.
 - Eligibility traces and $TD(\lambda)$.

Takeaways for this part of class

- Markov decision problems provide a general model of goal-oriented interaction with an environment.
- Reinforcement learning considers Markov decision problems where transition probabilities are unknown.
- A leading approach is based on estimating action-value functions.
- If state and action spaces are small, this can be done in tabular form, otherwise approximation (e.g., using neural nets) is required.
- We will distinguish between on-policy and off-policy learning.

Introduction

- Many interesting problems can be modeled as Markov decision problems.
- Biggest successes in game play (Backgammon, Chess, Go, Atari games,...), where lots of data can be generated by self-play.
- Basic framework is familiar from macro / structural micro, where it is solved using dynamic programming / value function iteration.
- Big difference in reinforcement learning:
Transition probabilities are not known, and need to be learned from data.
- This makes the setting similar to bandit problems, with the addition of changing states.
- We will discuss several approaches based on estimating action-value functions.

Markov decision problems

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Markov decision problems

- Time periods $t = 1, 2, \dots$
- States $\mathbf{S}_t \in \mathcal{S}$ (This is the part that's new relative to bandits!)
- Actions $\mathbf{A}_t \in \mathcal{A}(\mathbf{S}_t)$
- Rewards R_{t+1}
- Dynamics (transition probabilities):

$$P(\mathbf{S}_{t+1} = \mathbf{s}', R_{t+1} = r | \mathbf{S}_t = \mathbf{s}, \mathbf{A}_t = \mathbf{a}, \mathbf{S}_{t-1}, \mathbf{A}_{t-1}, \dots) = p(\mathbf{s}', r | \mathbf{s}, \mathbf{a}).$$

- The distribution depends only on the current state and action.
- It is constant over time.
- We will allow for continuous states and actions later.

Policy function, value function, action value function

- Objective: Discounted stream of rewards, $\sum_{t \geq 0} \gamma^t R_t$.
- Expected future discounted reward at time t , given the state $S_t = s$:
Value function,

$$V_t(s) = E \left[\sum_{t' \geq t} \gamma^{t'-t} R_{t'} \mid S_t = s \right].$$

- Expected future discounted reward at time t , given the state $S_t = s$ **and** action $A_t = a$:
Action value function,

$$Q_t(a, s) = E \left[\sum_{t' \geq t} \gamma^{t'-t} R_{t'} \mid S_t = s, A_t = a \right].$$

Bellman equation

- Consider a policy $\pi(\mathbf{a}|\mathbf{s})$, giving the probability of choosing \mathbf{a} in state \mathbf{s} . This gives us all transition probabilities, and we can write expected discounted returns recursively

$$Q_{\pi}(\mathbf{a}, \mathbf{s}) = (\mathcal{B}_{\pi}Q_{\pi})(\mathbf{a}, \mathbf{s}) = \sum_{s', r} p(s', r | \mathbf{s}, \mathbf{a}) \left(r + \gamma \cdot \sum_{a'} \pi(a' | s') Q_{\pi}(a', s') \right).$$

- Suppose alternatively that future actions are chosen optimally. We can again write expected discounted returns recursively

$$Q_{*}(\mathbf{a}, \mathbf{s}) = (\mathcal{B}_{*}Q_{*})(\mathbf{a}, \mathbf{s}) = \sum_{s', r} p(s', r | \mathbf{s}, \mathbf{a}) \left(r + \gamma \cdot \max_{a'} Q_{*}(a', s') \right).$$

Existence and uniqueness of solutions

- The operators \mathcal{B}_π and \mathcal{B}_* define contraction mappings on the space of action value functions. (As long as $\gamma < 1$.)
- By Banach's fixed point theorem, unique solutions exist.
- The difference between assuming a given policy π , or considering optimal actions $\operatorname{argmax}_a Q(\mathbf{a}, \mathbf{s})$, is the dividing line between **on policy** and **off policy** methods in reinforcement learning.

Markov decision problems

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Expected updates - dynamic programming

- Suppose we know the transition probabilities $p(\mathbf{s}', r | \mathbf{s}, \mathbf{a})$.
- Then we can in principle just solve for the action value functions and optimal policies.
- This is typically assumed in macro, IO models.
- Solutions: Dynamic programming.
Iteratively replace
 - $Q_\pi(\mathbf{a}, \mathbf{s})$ by $(\mathcal{B}_\pi Q_\pi)(\mathbf{a}, \mathbf{s})$, or
 - $Q_*(\mathbf{a}, \mathbf{s})$ by $(\mathcal{B}_* Q_*)(\mathbf{a}, \mathbf{s})$.
- Decision problems with terminal states: Can solve in one sweep of backward induction.
- Otherwise: Value function iteration until convergence – replace repeatedly.

Markov decision problems

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Sample updates

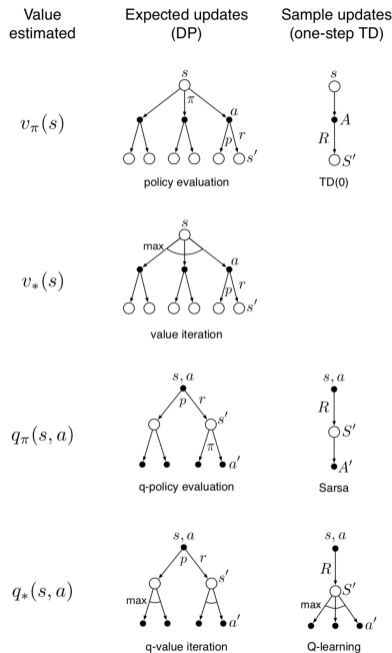
- In practically interesting settings, agents (human or AI) typically don't know the transition probabilities $p(\mathbf{s}', r | \mathbf{s}, \mathbf{a})$.
- This is where reinforcement learning comes in.
Learning from observation while acting in an environment.
- Observations come in the form of tuples

$$\langle \mathbf{s}, \mathbf{a}, r, \mathbf{s}' \rangle.$$

- Based on a sequence of such tuples, we want to learn Q_π or Q_* .

Classification of one-step reinforcement learning methods

1. Known vs. unknown transition probabilities.
2. Value function vs. action value function.
3. On policy vs. off policy.
 - We will discuss Sarsa and Q-learning.
 - Both: unknown transition probabilities and action value functions.
 - First: “tabular” methods, where we keep track off all possible values (\mathbf{a}, \mathbf{s}) .
 - Then: “approximate” methods for richer spaces of (\mathbf{a}, \mathbf{s}) , e.g., deep neural nets.



Sarsa

- On policy learning of action value functions.
- Recall Bellman equation

$$Q_{\pi}(\mathbf{a}, \mathbf{s}) = \sum_{s', r} p(s', r | \mathbf{s}, \mathbf{a}) \left(r + \gamma \cdot \sum_{a'} \pi(a' | s') Q_{\pi}(a', s') \right).$$

- Sarsa estimates expectations by sample averages.
- After each observation $\langle \mathbf{s}, \mathbf{a}, r, \mathbf{s}', \mathbf{a}' \rangle$, replace the estimated $Q_{\pi}(\mathbf{a}, \mathbf{s})$ by

$$Q_{\pi}(\mathbf{a}, \mathbf{s}) + \alpha \cdot (r + \gamma \cdot Q_{\pi}(\mathbf{a}', \mathbf{s}') - Q_{\pi}(\mathbf{a}, \mathbf{s})).$$

- α is the step size / speed of learning / rate of forgetting.

Sarsa as stochastic (semi-)gradient descent

- Think of $Q_\pi(\mathbf{a}, \mathbf{s})$ as prediction for $Y = r + \gamma \cdot Q_\pi(\mathbf{a}', \mathbf{s}')$.

- Quadratic prediction error:

$$(Y - Q_\pi(\mathbf{a}, \mathbf{s}))^2.$$

- Gradient for minimization of prediction error for current observation w.r.t. $Q_\pi(\mathbf{a}, \mathbf{s})$:

$$-(Y - Q_\pi(\mathbf{a}, \mathbf{s})).$$

- Sarsa is thus a variant of stochastic gradient descent.
- Variant: Data are generated by actions where π is chosen as the optimal policy for the current estimate of Q_π .
- Reasonable method, but convergence guarantees are tricky.

Q-learning

- Similar to Sarsa, but **off policy**.
- Like Sarsa, estimate expectation over $p(\mathbf{s}', r | \mathbf{s}, \mathbf{a})$ by sample averages.
- Rather than the observed next action \mathbf{a}' consider the optimal action **$\operatorname{argmax}_{\mathbf{a}'} Q_*(\mathbf{a}', \mathbf{s}')$** .
- After each observation $\langle \mathbf{s}, \mathbf{a}, r, \mathbf{s}' \rangle$, replace the estimated $Q_*(\mathbf{a}, \mathbf{s})$ by

$$Q_*(\mathbf{a}, \mathbf{s}) + \alpha \cdot \left(r + \gamma \cdot \operatorname{max}_{\mathbf{a}'} Q_*(\mathbf{a}', \mathbf{s}') - Q_*(\mathbf{a}, \mathbf{s}) \right).$$

Markov decision problems

Expected updates - dynamic programming

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Approximation

- So far, we have implicitly assumed that there is a small, finite number of states \mathbf{s} and actions \mathbf{a} , so that we can store $Q(\mathbf{a}, \mathbf{s})$ in tabular form.
- In practically interesting cases, this is not feasible.
- Instead assume parametric functional form for $Q(\mathbf{a}, \mathbf{s}; \theta)$.
- In particular: Deep neural nets!
- Assume differentiability with gradient $\nabla_{\theta} Q(\mathbf{a}, \mathbf{s}; \theta)$.

Stochastic gradient descent

- Denote our prediction target for an observation $\langle \mathbf{s}, \mathbf{a}, r, \mathbf{s}', \mathbf{a}' \rangle$ by

$$Y = r + \gamma \cdot Q_{\pi}(\mathbf{a}', \mathbf{s}'; \theta).$$

- As before, for the on-policy case, we have the quadratic prediction error

$$(Y - Q_{\pi}(\mathbf{a}, \mathbf{s}; \theta))^2.$$

- Semi-gradient: Only take derivative for the $Q_{\pi}(\mathbf{a}, \mathbf{s}; \theta)$ part, but not for the prediction target Y :

$$-(Y - Q_{\pi}(\mathbf{a}, \mathbf{s}; \theta)) \cdot \nabla_{\theta} Q(\mathbf{a}, \mathbf{s}; \theta).$$

- Stochastic gradient descent updating step: Replace θ by

$$\theta + \alpha \cdot (Y - Q_{\pi}(\mathbf{a}, \mathbf{s}; \theta)) \cdot \nabla_{\theta} Q(\mathbf{a}, \mathbf{s}; \theta).$$

Off policy variant

- As before, can replace \mathbf{a}' by the estimated optimal action.
- Change the prediction target to

$$Y = r + \gamma \cdot \max_{\mathbf{a}'} Q_*(\mathbf{a}', \mathbf{s}'; \theta).$$

- Updating step as before, replacing θ by

$$\theta + \alpha \cdot (Y - Q_*(\mathbf{a}, \mathbf{s}; \theta)) \cdot \nabla_{\theta} Q_*(\mathbf{a}, \mathbf{s}; \theta).$$

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Multi-step updates

- All methods discussed thus far are one-step methods.
- After observing $\langle \mathbf{s}, \mathbf{a}, r, \mathbf{s}', \mathbf{a}' \rangle$, only $Q(\mathbf{a}, \mathbf{s})$ is targeted for an update.
- But we could pass that new information further back in time, since

$$Q(\mathbf{a}, \mathbf{s}) = E \left[\sum_{t'=t}^{t+k} \gamma^{t'-t} R_{t'} + \gamma^{k+1} Q(\mathbf{A}_{t+k+1}, \mathbf{S}_{t+k+1}) \mid \mathbf{A}_t = \mathbf{a}, \mathbf{S}_t = \mathbf{s} \right].$$

- One possibility: at time $t+k+1$, update θ using the prediction target

$$Y_t^k = \sum_{t'=t}^{t+k-1} \gamma^{t'-t} R_{t'} + \gamma^k Q_\pi(\mathbf{A}_{t+k}, \mathbf{S}_{t+k}).$$

- k -step Sarsa: At time $t+k$, replace θ by

$$\theta + \alpha \cdot \left(Y_t^k - Q_\pi(\mathbf{A}_t, \mathbf{S}_t; \theta) \right) \cdot \nabla_{\theta} Q_\pi(\mathbf{A}_t, \mathbf{S}_t; \theta).$$

$TD(\lambda)$ algorithm

- Multi-step updates can result in faster learning.
- We can also weight the prediction targets for different numbers of steps, e.g. using weights λ^k :

$$Y_t^k = \sum_{t'=t}^{t+k} \gamma^{t'-t} R_{t'} + \gamma^{k+1} Q_{\pi}(A_{t+k+1}, S_{t+k+1}),$$

$$Y_t^{\lambda} = (1 - \lambda) \sum_{k=1}^{\infty} \lambda^k \cdot Y_t^k.$$

- But don't we have to wait forever before we can make an update based on Y_t^{λ} ?
- Note quite, since we can do the updating piece-wise!
- This idea leads to the so-called $TD(\lambda)$ algorithm.

Eligibility traces

- For $TD(\lambda)$, we proceed as for one-step Sarsa, using the prediction target

$$Y_t = R_t + \gamma \cdot Q_\pi(\mathbf{A}_{t+1}, \mathbf{S}_{t+1}; \theta).$$

- But we replace the gradient $\nabla_\theta Q_\pi(\mathbf{A}_t, \mathbf{S}_t; \theta)$ by a weighted average of past gradients, the so-called eligibility trace: Let $\mathbf{Z}_0 = \mathbf{0}$ and

$$\mathbf{Z}_t = \gamma\lambda \cdot \mathbf{Z}_{t-1} + \nabla_\theta Q_\pi(\mathbf{A}_t, \mathbf{S}_t; \theta).$$

- Updating step: At time t replace θ by

$$\theta + \alpha \cdot (Y_t - Q_\pi(\mathbf{A}_t, \mathbf{S}_t; \theta)) \cdot \mathbf{Z}_t.$$

- This exactly implements the updating by Y_t^λ in the long run.
- This is one of the most popular and practically successful reinforcement learning algorithms.

References

- *Sutton, R. S. and Barto, A. G. (2018). Reinforcement learning: An introduction. MIT press.*
- *François-Lavet, V., Henderson, P., Islam, R., Bellemare, M. G., and Pineau, J. (2018). An introduction to deep reinforcement learning. Foundations and Trends® in Machine Learning, 11(3-4):219–354.*