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Corrigendum Instrumental Variables with Unrestricted Heterogeneity and Continuous Treatment

STEFAN HODERLEIN
Boston College
HAJO HOLZMANN
University of Marburg
MAXIMILIAN KASY
Harvard University
and
ALEXANDER MEISTER
University of Rostock

Kasy (2014) considers a triangular system of equations characterized by the following assumptions:

Assumption 1. (Triangular system).

$$Y = g(X, U)$$
$$X = h(Z, V) \tag{1}$$

where X, Y, Z are random variables taking their values in $\mathbb{R}$, the unobservables U, V have their support in an arbitrary measurable space of unrestricted dimensionality, and

$$Z \perp (U, V). \tag{2}$$

Assumption 2. (Continuous treatment). *The treatment X is continuously distributed in $\mathbb{R}$ conditional on Z.*

Assumption 3. (First stage monotonic in instrument). *The first stage relationship $h(z,v)$ is strictly increasing in z for all v.*

Assumption 4. (Continuous instrument). *The instrument Z is continuously distributed in $\mathbb{R}$, with support $[z_l, z_u]$. The first stage relationship h is continuous in z for all z and almost all v, and $P(X \leq x|Z=z)$ is continuous in z for all x.*

Under these assumptions, the following definitions are introduced:

Definition 1. (Potential outcomes). *We denote by*

$$\begin{aligned} Y^x &= g(x,U) \\ X^z &= h(z,V). \end{aligned} \tag{3}$$

Furthermore, we define

$$Z^x = \begin{cases} h^{-1}(x,V) & \text{if } h(z_l,V) \leq x \leq h(z_u,V) \\ -\infty & \text{if } x < h(z_l,V) \\ \infty & \text{if } h(z_u,V) < x. \end{cases} \tag{4}$$

It is claimed in the statement and proof of Kasy (2014), theorem 1, that under these assumptions:

$$\begin{aligned} P(Y \leq y|X=x,Z=z) &= P(Y^x \leq y|X=x,Z=z) \\ &= P(Y^x \leq y|Z^x=z,Z=z) \\ &= P(Y^x \leq y|Z^x=z). \end{aligned}$$

This assertion is wrong, the following theorem 1 states a corrected version. For theorem 1 to hold, we need to additionally impose the following regularity conditions.

Assumption 5. (Regularity conditions). *There exist $0 < c_l < c_u < \infty$, such that $c_l \leq \partial_z h(z,v) \leq c_u$ for all z and v. Further, V can be decomposed as $V=(V_1,V_2)$, where V_2 is scalar and absolutely continuous given (Z,V_1) with bounded conditional density, and $\partial_{v_2} h(z,v_1,v_2) \geq c > 0$ for all z and $v=(v_1,v_2)$.*

The second part of assumption 5 ensures that X^z is continuously distributed for all z with a density that is bounded from above.

Theorem 1. *Under assumptions 1 through 5,*

$$P(Y \leq y|X=x,Z=z) = E[\lambda^{x,z} \cdot \mathbf{1}(Y^x \leq y)|Z^x=z] \tag{5}$$

where

$$\lambda^{x,z} = \frac{E[\partial_z h(z,V)|X^z=x]}{\partial_z h(z,V)} = \frac{\left[\partial_z h(z,V)\right]^{-1}}{E[[\partial_z h(z,V)]^{-1}|Z^x=z]}. \tag{6}$$

Theorem 1 immediately implies the following two corollaries. Corollary 1 considers the control function approach of Imbens and Newey (2009), using the control function $V^* = F(X|Z)$. Corollary 1 provides a representation as a weighted average for the estimand for the average structural function proposed by Imbens and Newey (2009). Corollary 2 establishes that the claims of Kasy (2014) *do* hold if the additional assumption is imposed that first stage heterogeneity V is one-dimensional.

Corollary 1. *Let* $V^* = F(X|Z)$*, and assume* $\text{supp}(V^*|X=x) = [0,1]$*. Then*

$$\int_0^1 E\big[Y|X=x, V^*=v^*\big]dv^* = E\Big[\lambda^{x,Z^x}\cdot Y^x\Big].$$

Corollary 2. *If* $\dim(V)=1$ *(ie., $V=V_2$) and h is strictly monotonic in V, then*

$$P(Y \le y|X=x, Z=z) = E[\mathbf{1}(Y^x \le y)|Z^x = z]. \tag{7}$$

Discussion The key step which fails in the original derivation of Kasy (2014) is the asserted equality

$$P(Y^x \le y|X=x, Z=z) = P(Y^x \le y|Z^x = z, Z=z).$$

The conditioning event on both sides is the same, that is

$$(X=x, Z=z) = (Z^x = z, Z=z).$$

If this conditioning event had a positive probability, as would be the case for discrete random variables, the asserted equality would indeed hold. As we are dealing with the continuous case, however, this event has probability zero. Conditional expectations (probabilities) given events of probability zero are only well defined relative to a given σ-algebra. Since the σ-algebra generated by the random variables (X,Z) and the σ-algebra generated by the random variables (Z^x, Z) are different, equality of conditional distributions need not hold in general.

As implied by corollary 2, the assertions of Kasy (2014) *do* hold under the assumption imposed by Imbens and Newey (2009), that first stage heterogeneity V is one-dimensional and enters h monotonically. Despite the failure of its central theorem to hold, Kasy (2014) might thus be thought of as providing alternative estimation procedures, based on reweighting rather than based on regression with controls, which are valid under the assumptions of Imbens and Newey (2009). An alternative approach to identification in triangular systems that avoids first stage scalar monotonicity is discussed by Hoderlein *et al.* (2016), who consider the random coefficient case.

Proof of theorem 1. This proof is structured as follows. We first consider the right hand side of equation (5), and show that for non-negative random variables ϕ such that $(\phi, V) \perp Z$, we get:

$$E[\phi|Z^x = z] = \frac{E[\phi \cdot \partial_z h(z,V)|X^z = x]}{E[\partial_z h(z,V)|X^z = x]}. \tag{8}$$

We then turn to the left-hand side, and show, for ψ such that $(\psi, V) \perp Z$,

$$E[\psi|X=x, Z=z] = E[\psi|X^z = x].$$

The claim of the theorem then follows once we consider $\psi = \mathbf{1}(Y^x \le y)$ and $\phi = \lambda^{x,z}\cdot\mathbf{1}(Y^x \le y)$.

Consider some non-negative random variable ϕ, defined on the same probability space as V and Z, such that $(\phi, V) \perp Z$ and $0 < E[\phi] < \infty$. Since

$$\begin{aligned}\partial_z E[\phi \cdot \mathbf{1}(Z^x \le z)] &= \partial_z \int_{-\infty}^{z} E[\phi|Z^x = z']\cdot f_{Z^x}(z')dz' \\ &= E[\phi|Z^x = z]\cdot f_{Z^x}(z) \end{aligned} \tag{9}$$

and

$$\partial_z E[\mathbf{1}(Z^x \leq z)] = \partial_z \int_{-\infty}^{z} f_{Z^x}(z')dz' \\ = f_{Z^x}(z),$$

we can write:

$$E[\phi | Z^x = z] = \frac{\partial_z E[\phi \cdot \mathbf{1}(Z^x \leq z)]}{\partial_z E[\mathbf{1}(Z^x \leq z)]} = \frac{\partial_z E[\phi \cdot \mathbf{1}(h(z,V) \geq x)]}{\partial_z E[\mathbf{1}(h(z,V) \geq x)]}. \tag{10}$$

The second equality holds by monotonicity of h, which implies $Z^x \leq z$ if and only if $X^z = h(z,V) \geq x$. Let us first consider the denominator of equation (10). We get:

$$\begin{aligned} \partial_z E[\mathbf{1}(h(z,V) \geq x)] &= \partial_z E[\mathbf{1}(X^z \geq x)] \\ &= -\partial_z F_{X^z}(x) \\ &= E\left[\partial_z h(z,V) | X^z = x\right] \cdot f_{X^z}(x). \end{aligned}$$

These equalities hold (1) by definition of the pdf $f_{Z^x}(z)$, (2) by the equality $\mathbf{1}(Z^x \leq z) = \mathbf{1}(X^z \geq x)$ (due to monotonicity of h), and (3) by equation (D1) in Chernozhukov *et al.* (2015) (see also Hoderlein and Mammen, 2007). This last step requires the regularity conditions of assumption 5.

Let us now turn to the numerator of equation (10). Consider the probability measure P^ϕ, defined by $\partial P^\phi / \partial P = \phi / E[\phi]$, that is, the probability measure with density $(\phi / E[\phi])$ relative to P, and let E^ϕ be the expectation operator with respect to P^ϕ. Applying the same reasoning as before to this new measure yields the numerator of equation (10),

$$\begin{aligned} \frac{1}{E[\phi]} \cdot \partial_z E[\phi \cdot \mathbf{1}(h(z,V) \geq x)] &= \partial_z E^\phi[\mathbf{1}(X^z \geq x)] \\ &= -\partial_z F^\phi_{X^z}(x) \\ &= E^\phi\left[\partial_z h(z,V) | X^z = x\right] \cdot f^\phi_{X^z}(x) \\ &= \tfrac{1}{E[\phi]} \cdot E\left[\phi \cdot \partial_z h(z,V) | X^z = x\right] \cdot f_{X^z}(x). \end{aligned}$$

The last equality holds by the general properties of Radon–Nikodym derivatives, since, by the same argument as in equation (9),

$$f^\phi_{X^z}(x) = f_{X^z}(x) \cdot \tfrac{1}{E[\phi]} E[\phi | X^z = x],$$

and

$$E^\phi\left[\partial_z h(z,V) | X^z = x\right] = \frac{E[\phi \cdot \partial_z h(z,V) | X^z = x]}{E[\phi | X^z = x]}.$$

The claim of equation (8) follows from what we have shown so far. This proves our first assertion, and also implies the equality of the two definitions of $\lambda^{x,z}$ (set $\phi = 1/\partial_z h(z,V)$) given in the statement of the theorem.

Let us now turn to the left hand side of the equality asserted in the theorem. Consider some random variable ψ, again defined on the same probability space as V and Z, such that $(\psi, V) \perp Z$ and $E[|\psi|] < \infty$. Using statistical independence of Z and (ψ, V), we get:

$$\begin{aligned} E[\psi | X = x, Z = z] &= E[\psi | h(z,V) = x, Z = z] \\ &= E[\psi | h(z,V) = x] = E[\psi | X^z = x]. \end{aligned}$$

Setting $\psi = \mathbf{1}(Y^x \leq y)$ and $\phi = \lambda^{x,z} \cdot \mathbf{1}(Y^x \leq y)$ concludes the proof. ‖

Proof of corollary 1. This follows immediately from Theorem 1 and

$$\int_0^1 E\big[Y|X=x, V^*=v^*\big]dv^* = \int_{-\infty}^{\infty} E[Y|X=x, Z=z]dF_{Z^x}(z),$$

cf. Kasy (2014), proof of theorem 2. ‖

Proof of corollary 2. Under this condition, V is pinned down by $Z^x = z$, so that $\lambda^{x,z} \equiv 1$ in view of $\lambda^{x,z} = \big[\partial_z h(z,V)\big]^{-1} / E[[\partial_z h(z,V)]^{-1} | Z^x = z]$. ‖

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